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Boolean neural nets are observable

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Abstract

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It is shown that arbitrary locally finite discrete neural networks are observable (have the shadowing property) in the sense that pseudo-orbits obtained by small perturbations of an orbit are approximated by actual orbits. The model includes discretizations of analog networks, arbitrary cellular automata, and a wide generalization of linear maps on a one dimensional grid. It follows that the true qualitative behavior of dynamical systems can be observed to infinite precision on computer simulations, despite unavoidable discretization and approximation errors.

1. Introduction

We explore aspects of the general problem of simulation of dynamical systems via parallel computers, as embodied by fine-grained parallel models of computation such as cellular automata (CA) and neural networks (NN). They can be regarded as discrete dynamical systems that provide an unusual combination of complex dynamical behavior out of very simple local exchanges in a network of parallel processors. This evolution can be observed in computer simulations, which at each step might introduce negligible errors that conceivably accumulate under iteration. It is quite true that, in general, a computer simulation of an orbit of a given continuous system (known as a pseudo-orbit) is, in fact, far from the orbit in the real system. Systems satisfying the shadowing property are systems for which pseudo-orbits are uniformly approximated by real orbits so that the long term orbital behavior is to some extent captured by pseudo-orbits. This was the motivation of the well-known shadowing property. Pseudo-orbits also arise as trajectories of simple dynamical processes

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obtained by computer simulation. In such cases, especially when errors propagate exponentially, it is important to know when the numerical process is actually approximated by a trajectory of the real process. However, it appears difficult to give a characterization of exactly which maps on the interval possess the property (see e.g. [3, 4]).

Roughly speaking, a CA consists of a lattice of sites labelled with a finite set of states, evolving according to some local rule. The phase space of a CA on an infinite grid is a totally disconnected compact metric space. The global map of such a system on a totally disconnected space is known to be characterized by continuity and commutation with the basic shifts under a particular encoding of the space. CA can thus be regarded as models of parallel computers where local rules play the role of a common computer program for each processor. They can be readily generalized to Boolean neural networks consisting of less homogeneous grids of parallel processors with local rules of an add-squash type that may vary from site to site. In either case, the evolution of the system corresponds to running the program(s) on initial data (conditions). The shadowing property thus seems to be a desirable dynamical property of parallel systems given by local programs from the point of view of computation.

We show that several conditions imply the shadowing property in the state space of neural networks. As in the Euclidean space case, giving a characterization of exactly which maps possess this property appears to be an interesting open problem. We begin with the most basic question about shadowing on these spaces, namely whether the identity map has the property. Unlike the situation with real spaces, the identity does, and, as a matter of fact, it characterizes totally disconnected compact spaces. We then prove that if their (global) dynamics are toggle or linear, they have the shadowing property. It follows that maps induced by (possibly infinite) boolean networks all have the property. Finally we prove that, in particular, all types of cellular automata also have the property.

We finish this section with some notation and precise definitions. Let $f: X \to X$ be a continuous map of a compact metric space with metric $|*, *|$. The *orbit* of $x \in X$ is the sequence $\{x, f(x), \ldots, f^n(x) \ldots\}$. A sequence $\{x_n\}_{n \geqslant 0}$ is an orbit if and only if $f(x_n) = x_{n+1}$, for $n \geqslant 0$. Given a number $\delta > 0$, a *δ-pseudo-orbit* is a sequence $\{x_n\}$ so that the distances $|f(x_n), x_{n+1}| < \delta$ for all $n \geqslant 0$.

Definition 1.1. The map f is observable (or has the shadowing property) if and only if for any $\varepsilon > 0$ there exists a $\delta > 0$ so that any δ-pseudo-orbit $\{x_n\}$ is ε-approximated by the orbit, under f, of some point $z \in X$ i.e.,

$$|x_n, f^n(z)| < \varepsilon, \quad \text{for all } n \geqslant 0.$$

If S is a finite nonempty set (the possible states of each cell), a *configuration* is a mapping $x: Z \to S$. A configuration is best thought of as a bi-infinite sequence with entries from S denoted x_i, although it may also describe a similar map with domain the integer grid $\mathbf{Z}^n$ (or even more general infinite graphs) under some enumeration of its vertices. The set of all integers will be used to name the sites of any one-dimensional bi-infinite sequence. The space $\mathbf{C}$ of all configurations is a compact metric space if endowed with any of a number of metrics, such as $|x, y| := (1/2^n)$, where $n = \inf\{|i| : x_i \neq y_i\}$. In this metric, a small perturbation of a configuration is obtained by changing the values of x at pixels far away from x_0.

A cellular automaton of radius r is a map T completely determined by a local rule (or table) $\delta : S^{2r+1} \to S$ that preserves the zero configuration and commutes with the shift. The image of a configuration x is given by

$$T(x)_i := \delta(x_{i-r}, \ldots, x_i, \ldots, x_{i+r}).$$

Discrete neural networks are generalizations of cellular automata where the state set is endowed with an addition and multiplication and the local rule is determined by "squash" function f_i (typically a threshold functions) which may vary from site to site. Examples of cellular automata are left and right shifts and many more can be found in [17, 14]. It is well known that such a mapping T is a continuous transformation of $\mathbf{C}$. In fact, the Hedlund–Richardson's theorem [10, 13] states that this property together with shift-commutation characterize global maps induced by local rules.

2. Shadowing of the identity

Bowen showed that sufficiently hyperbolic systems in real Euclidean spaces do have the shadowing property [3]. The simplest question about shadowing concerns the shadowing property of the identity function. Surprisingly, the identity map defined on a closed interval $[a, b]$ does not have the shadowing property. In fact, just consider $\varepsilon := |b-a|/4$. For no $\delta > 0$ is there an orbit that ε-approximates the pseudo-orbit $\{x_n\}$ defined by $x_n := a + n\min\{\delta/2, |b-a|/4\}$ if $a + n\min\{(\delta/2, |b-a|/4\} \leqslant b$, and b otherwise.

In this section we prove a characterization of totally disconnected spaces via the shadowing condition on the identity map. The result is also interesting in theoretical studies of massively parallel computers (i.e. cellular automata and neural networks [9]) and their applications to computation of real-valued objects [6].

Theorem 2.1. *The identity map* **id** *of a compact metric space X has the shadowing property iff X is totally disconnected.*

Proof. Assume X is not totally disconnected. Let Ω be a connected component with positive diameter of a point $a \in X$. Since the closure of a connected set is connected, Ω is closed. Choose two distinct points $x, y \in \Omega$ and put $\varepsilon := |x, y|/4$. For any $\delta > 0$ and any two points $\alpha, \beta \in \Omega$, there exists a finite chain of open balls $B_1, \ldots, B_k$ of radii $\delta_0 > 0$, where $\delta_0 := \min\{\varepsilon/4, \delta/2\}$, such that $\alpha \in B_1$, $\beta \in B_k$ and $B_i \cap B_{i+1} \neq \Phi$ for any two consecutive balls B_i, B_{i+1}. Let $z_i \in B_i \cap B_{i+1}$. Obviously, $x, z_1, z_2, \ldots, z_{k-1}, y, y, \ldots$ is a pseudo-orbit which is not ε-traceable.

For the sufficiency, let $\varepsilon > 0$ and $\{B(x, \varepsilon/2) : x \in X\}$ be a covering of X. For each x there exists a closed and open subset C_x such that $x \in C_x \subset B(x, \varepsilon/2)$ since $B(x, \varepsilon/2)$ is disconnected. Since $\{C_x : x \in X\}$ is a covering of X, there exists a finite subcover $C_{x_1}, \ldots, C_{x_k}$ of closed and open subsets. Now define

$$U_1 := C_{x_1}, \qquad U_2 := C_{x_2} - C_{x_1}, \ldots, \qquad U_k := C_{x_k} - \left(\bigcup_{j<k} C_{x_j} \right).$$

The sets $U_1, \ldots, U_k$ are open, closed, disjoint, and cover X. Let $0 < \delta := \min\{\varepsilon, |U_i, U_j| : i \neq j\}$. Every δ-pseudo-orbit $\{x_n\}$ of **id** lies entirely in one (and only one) C_{x_j}. Since

$$diameter(C_{x_j}) < \varepsilon,$$

for any point $y \in C_x$, $|x_n, y| < \varepsilon$, as required.

Note that X being totally disconnected is, moreover, necessary and sufficient for **id** to have the asymptotic shadowing property (where pseudo-orbits are traceable only asymptotically). The pseudo-orbit

$$x, z_1, z_2, \ldots, z_{k-1}, y, z_{k-1}, \ldots, z_1, x, z_1, z_2, \ldots, \ldots$$

is not asymptotically traceable (with x at the appropriate position depending on the threshold N). $\square$

3. Toggle rules and the extension property

Cellular automata in higher dimensions can be analyzed as one-dimensional rules which are not necessarily local and/or homogeneous. Generalizations of this type have been considered in [8, 9, 5] in the form of neural and automata networks.

Definition 3.1. Let T be a global map of configuration space in the line. A site j T-influences site i if there exist $x, y \in \mathbf{C}$ such that $x_l = y_l$ for all $l \neq j$ but $T(x)_i \neq T(y)_i$. The neighborhood of a site i is the set of indices

$$N_i(T) := \{ j : j \text{ } T\text{-influences } i \}.$$

(Reference to T will be dropped if T is clear from context.) A rule is strictly local if $N_i \subseteq \{i\}$ for every site i.

Example 1. A discrete neural network is given as follows. The vertices of a digraph (V, A) hold the sites, the state set S is endowed with addition and multiplication, and there is an *activation function* f_i for each site i. The global map T is given by

$$T(x)_i := f_i\left[\sum_{j\in N_i} a_j x_j\right],$$

where $N_i := \{j : ji \in A\}$ is the subset of sites with "synaptic connections" w_{ji} into i. Neural networks are a generalization of cellular automata [5, Theorem 2].

Example 2. It will be shown in Section 4.1 that any higher dimensional cellular automaton can be regarded as nonhomogeneous discrete neural network *on the line* with neighborhoods uniformly bounded in size.

In this paper we only consider locally finite maps T for which all the N_i's are finite.

Lemma 3.1. *Every strictly local CA has the shadowing property.*

Proof. Since there is no interaction across sites, every ε-pseudo-orbit can be ε-traced by the orbit of the first element. □

The following notion appears in a particular form in [10] under the name *permutive*. Hedlund calls a cellular automaton *permutive* (other authors call it *toggle*) at position j if the local rule defines a permutation (bijection) of the state set when the arguments at all positions other than j are arbitrarily fixed. Some notation is needed for the generalization. In a compact metric space X, let $YB[y, \varepsilon]$ denote the closed ball of center $y \in Y$ and radius ε in the subspace Y with the induced topology. (Y is omitted if $Y = X$.) The i-range $R(T)_i$ of a global map T is the projection of the range $R(T)$ of T to site i. The *product range* of T, denoted P, is the Cartesian product $\prod_i R(T)_i$ of all i-ranges of T.

Definition 3.2. A map T is i-toggle at position $j \in N_i$ iff the restriction of T to site j is surjective onto $R(T)_i$ for arbitrarily fixed assignments of the values of sites outside of site j. The map T is toggle iff T is i-toggle at some site from each of the sets

$$N_i^* - \bigcup_{j<i} N_j \quad \text{and} \quad N_i^* - \bigcup_{j>i} N_j, \tag{1}$$

where N_i^* denotes $N_i - \{i\}$. The neighborhoods N_i are said to be left-threaded (respectively right-threaded, threaded) if the first (second, both) neighborhood condition(s) (1) hold(s) for every site i.

Examples. The XOR rule on any number of neighbors in binary is i-toggle at any position. The left-shift is i-toggle at position $i+1$ in one-dimensional Euclidean CA, but not at any other position. In fact, every binary linear rule is toggle at positions on which the next-state of the center cell effectively depends. Later, we will use the fact that the Cartesian product of two toggle rules is toggle, as can be easily checked.

In order to prove that toggle maps have the shadowing property we make use of the following result, which is closely related (in fact, it follows by a similar proof) to a sufficient condition for shadowing due to Coven–Kan–Yorke and contained in [4, Lemmas 2.3–2.4].

Lemma 3.2. *Let X be a compact metric space and T a continuous map on X. If for every ε, $\delta>0$ the map T on X satisfies*

$$PB[T(x), \varepsilon+\delta] \subseteq T(B[x, \varepsilon]), \tag{2}$$

then any δ-pseudo-orbit can be ε-traced by an orbit of T. In particular, if for every ε there exists such a δ, T has the shadowing property.

A *pseudoblock* is the restriction of a one-dimensional configuration x to a set of sites (called its *support* and denoted $\underline{B}$). It is a *block* if the sites are finite in number and contiguous. There is an obvious operation of concatenation between pseudoblocks B, B' with disjoint supports, simply denoted BB', whose support is the union of the supports of B and B'. The action of T restricts to pseudoblocks B as follows. The image $T(B)$ is a pseudoblock with support $\{i: N_i \subseteq \underline{B}\}$ so that $T(B)_i := T(x)_i$, where x is any configuration such that x agrees with B on its support. In this case, the pseudoblock $T(B)$ is said to be in the range of T.

Definition 3.3. A global map T satisfies the extension property (EP) if for every pair of pseudoblocks B, D with the supports of $T(B)$ and D disjoint, there exists a pseudoblock B' whose support is disjoint from $\underline{B}$ such that

$$T(BB') = T(B)D.$$

The EP is said to hold on blocks (EPB) if this property holds only where $T(B)$, D and $T(B)D$ are blocks.

Proposition 3.1. *Every toggle map has the extension property on blocks.*

Proof. By induction on the cardinality of $\underline{D}$. If $\underline{D}=\{k\}$, is an extra cell to the right of $\underline{T(B)}$, by toggleness, T is k-toggle at some cell $i \in N_k^* - \bigcup_{j<k} N_j$. Therefore there exists an extension of the pseudoblock B as desired. The inductive step follows likewise. □

Given a map T with state S, one can define another map T' with state set $S \times S$, called *the second-order rule* associated with T, by

$$T'\begin{pmatrix} x \\ y \end{pmatrix}_i = \begin{pmatrix} T(x)_i \\ x_i \end{pmatrix}.$$

Lemma 3.3. *A map has shadowing if its second-order map has shadowing.*

Proof. The metric in configuration space over $S\times S$ is the product metric. Given a δ-pseudo-orbit $\{x_n\}_{n\geqslant 0}$ for T, the sequence $\{(\binom{x_n}{x_{n-1}})\}_{n\geqslant 0}$ is a δ-pseudo-orbit of the second-order map, say traced by $\binom{z_0}{z_0}$. The orbit of z_0 traces the given pseudo-orbit. $\square$

Note that, for proofs of shadowing, the same technique can be used to prove that it can always be assumed, without loss of generality, that $i\in N_i$ for every i, and more generally, that $k\in N_i$ for any k, i.

Proposition 3.2. *Every toggle map has the shadowing property.*

Proof. It is sufficient to verify condition (2). By Lemma 3.3, assume that $i\in N_i$ for every i. For given $\varepsilon>0$, the condition is verified with $\delta:=\varepsilon$ for the distance given by $1/2^n$, where n is the largest positive integer such that $x_{[-n,n]}=y_{[-n,n]}$. Observe that this distance decreases exponentially as x, y agree on larger site blocks. Let $u\in PB[T(x), 2\varepsilon]$ agree with $T(x)$ on a maximal block B with support $[-n, n]$. Let $\underline{B}^-$ denote the pseudoblock obtained by adding to $\underline{B}$ all sites which influence a site in B. Let $D:=u_{[-n-1,n+1]}$. By toggleness, there exists a pseudoblock B' that coincides with x in the whole interval $[-n-1, n+1]$ whose image is D. The extension property now guarantees a configuration y that coincides with x on $[-n-1, n+1]$ and such that $T(y)=u$. Clearly $|x, y|\leqslant\varepsilon$, as desired. $\square$

4. Shadowing of linear maps

If the set S of admissible states is endowed with a binary operation,[1] the operation can be extended componentwise to the entire configuration space, herein denoted by simple concatenation. A self-map of configuration space is *linear* if it satisfies the superposition principle under this operation, i.e. if

$$\forall x, y\in \mathbf{C},\quad T(xy)=T(x)T(y).$$

In fact, for such a state set and operation, the superposition principle holds iff the operation is associative and commutative [12]. In this section we prove that linear maps in a much broader class than cellular automata satisfy the shadowing property using condition (2).

In order to simplify the general proof we map higher dimensional grids to the 1D grid in any of many known ways. Of necessity the encoding does not preserve locality, but it preserves linearity. This mapping destroys locality in the sense that the next-state of a site in the line depends on sites that are arbitrarily far away but

[1]For instance, an Abelian group structure which makes it a modulo over the ring of integers.

correspond to neighboring sites in the plane. However, they can be still called neighbors in the sense of definition 3.1.

Theorem 4.1. *Every linear continuous map has the shadowing property.*

We remark that there are linear maps without shadowing (of course, not continuous). Consider $T(x) := \mathbf{0}$ for all basis elements $x \neq \mathbf{1}$ in a(n uncountable) basis of $\mathbf{C}$ containing $\mathbf{1}$, and put $T(\mathbf{1}) := \mathbf{1}$, where $\mathbf{1}$ (respectively, $\mathbf{0}$) denotes the all-ones (all-zeroes) configuration in binary. The pseudo-orbit $\mathbf{1}, \mathbf{1}, \mathbf{1}', \mathbf{0}, \mathbf{0} \ldots$ cannot be 1/16-traced if the $'$ indicates a small perturbation of $\mathbf{1}$ with finite support.

Theorem 4.1 follows from the following intermediate results. First, we prove the following proposition.

Proposition 4.1. *Every linear continuous map with uniformly bounded neighborhoods satisfies condition* (2).

Second, continuity of linear maps is characterized as follows. (The result holds for arbitrary maps of configuration space – see the proof of Theorem 3.3 in [9].)

Proposition 4.2. *A linear map T is continuous iff every N_i is finite.*

In general, the shadowing property is not preserved under uniform limits. Still, this statement is true for the maps involved in the proof of the following proposition.

Proposition 4.3. *Every locally finite (respectively, linear) map is a uniform limit of (linear) maps with uniformly bounded neighborhoods.*

Proof of Proposition 4.1. First, by using the technique of Lemma 3.3, we can modify all neighborhoods of a given linear continuous maps so they are threaded. Second, it suffices to prove the result when the number of states is a prime power since every finite Abelian group is the direct sum of Abelian groups of prime power order. A linear map T thus splits into a Cartesian product $T' \times T''$ of linear maps which is obviously toggle if each T' and T'' are toggle.

We first prove the result for a prime number of states by contradiction. Assume T is not i-toggle and let $k \in N_i$ and a pseudoblock B with support $N_i^k := N_i - k$ be such that $T(B)_{ki}: S \to S$ is not surjective for some choice of states in N_i^k, i.e. $T(B)_{ki}(r) = T(B)_i(s)$ with $s \neq r$. By superposition of S, $T(O)_i(r-s) = 0$. Since $r-s$ is a generator of S, $T(B)_{ki}$ is constantly 0 when N_i^k is all 0. This contradicts that k influences i, and hence, T is toggle.

Lastly, for a prime power number of states the proof is similar. We illustrate the procedure with p^2 states. If T is as in the previous paragraph and $r-s$ is not a generator of $\mathbf{Z}_{p^2}$, the range of $T(B)_{ki}$ has order p since every element in $\mathbf{Z}_{p^2}$ is of the form $q(r-s)+d$, where d is a remainder between 0 and $p-1$. From linearity it follows that for every pseudoblock D the range $T(D)_{ik}$ has order p as well, hence T is i-toggle at k, a contradiction. □

Proof of Proposition 4.2. We use that T is continuous iff whenever a sequence $\{x^n\}$ converges to x, the sequence $T(x^n)$ also converges to $T(x)$. If N_i contained an infinite sequence of sites $\{j_n\}$, the sequence given by $x^n := \sum_{k \leqslant j_n} e^{k_j}$ (where e^j denotes the configuration given by $e_i^j := \delta_{ij}$) would converge to some configuration x, but the sequence $\{T(x^n)_i\}$ would not have a limit since $T(e^{j_n})$ is constantly adding nonzero terms. Therefore N_i must be finite. The converse is an obvious consequence of the nature of the product topology. □

Proof of Proposition 4.3. For the given map T and $n \geqslant 1$, consider the rules T_n with neighborhood $N_i(T_n) := [i-n, i+n]$ given by $T_n(x)_i := T(x_{[i-n,\, i+n]})_i$, where $x_{[i-n,\, i+n]}$ is obtained from x by replacing the values of x with 0 outside of $[i-n, i+n]$. Note that $|N_i(T_n)| \leqslant 2n+1$. Because T is locally finite, eventually $N_i(T) = N_i(T_n)$ for all i. Therefore, $|T(x), T_n(x)|$ becomes arbitrarily small uniformly on x, as $n \to \infty$. Under these conditions, it is clear that the T_ns are linear if and only if T is linear. Furthermore, it will be important to notice that the T_n satisfy condition (2) by Proposition 4.1 and the proof of Proposition 3.2. □

Proof of Theorem 4.1. We establish condition (2) for T since the maps T_n of the previous proof satisfy it as well. Given $\varepsilon > 0$, put $\delta := \varepsilon$. For $y \in PB[T(x), 2\varepsilon]$, the construction in the previous proof shows that y is the limit of a sequence $\{y^n\}$ of configurations in $PB[T_n(x), 2\varepsilon]$. By condition (2) for T_n, there exists configurations $z_n \in B[x, \varepsilon]$ such that $T_n(z_n) = y_n$. Let z be an accumulation point of $\{z_n\}$. Since

$$y = \lim y_n = \lim T_n(z_n) = T(z),$$

it follows that $y \in T(B[x, \varepsilon])$. □

5. Shadowing of neural networks

We can finally state and prove the main result of this paper.

Theorem 5.1. *Every neural network is observable, i.e. has the shadowing property.*

For the proof we need the following result which is a slight variation of a result in [9].

Theorem 5.2. *A continuous self-map of configuration space is induced by a locally finite neural network if and only if*

(1) *T is continuous;*
(2) *The support of every pixel's image $T(e^j)$ is finite;*
(3) *T is the composition $T = F \circ L$, where F is strictly local and L is linear.*

Proof of Theorem 5.1. Let F and L be maps as in Theorem 5.2 for a given neural network T and let L_n be a sequence for L as in the proof of Proposition 4.3. Since F is strictly local

$$PB[F(x), \varepsilon] \subseteq F(PB[x, \varepsilon]).$$

Since the L_ns satisfy condition (2) it follows that,

$$PB[F \circ L(x), 2\varepsilon] \subseteq F \circ L(PB[x, \varepsilon]).$$

Now it is easy to see that condition (2), and hence the shadowing property, holds for $F \circ L$, as well. □

Since cellular automata are particular cases of neural networks [5], it follows in particular that arbitrary cellular automata on Cayley graphs [5] have the shadowing property.

Theorem 5.3. *Every cellular automaton is observabe, i.e., has the shadowing property.*

In closing, we remark that it is possible to construct an infinite number of topologically inequivalent continuous self-maps of a totally disconnected space without the shadowing property. Let $I_n := (a_n, b_n)$ $(n \geq 1)$ be the sequence of intervals successively deleted in the construction of the standard ternary Cantor set $\mathscr{C}$ with $0 < a_n < b_n < a_{n+1} < b_{n+1}$ for all $n \geq 1$ so that the sequences $\{a_n\}$, $\{b_n\}$ converge to $\frac{1}{2}$. Let $\mathscr{C}_n := [b_n, a_{n+1}] \cap \mathscr{C}$ $(n \geq 0)$ be the trace of the Cantor set in the given intervals, where $b_0 := 0$. Since every two totally disconnected, compact, perfect and Hausdorff spaces are homeomorphic [11], let h be the map of $\mathscr{C}$ mapping $\mathscr{C}_0$ onto $\mathscr{C}_0 \cup \mathscr{C}_1$, $\mathscr{C}_n$ onto $\mathscr{C}_{n+1}$, and leaving 1 as a fixed point. The map f_1, obtained by juxtaposing a copy of h on the first half of the unit interval and a copy of h^{-1} on the second half, is a homeomorphism with one fixed point which is an attractor on the left and a repellor on the right, hence without the shadowing property. Similar maps f_n with exactly n fixed points can be constructed for every $n \geq 1$.

6. Conclusion and open problems

The dynamical behavior of discrete neural networks are observable to any degree of accuracy despite errors from a variety of sources.

They include, for example, approximation errors due to discretization and hardware implementation of an infinite number of units on a discrete computer with only finitely many neurons. The results in this paper raise the question whether analog neural networks with real-valued activations in discrete or continuous time are observable as well. This issue has been examined recently in the literature in various forms. First, analogous results about the question of identifiability (of weights for a given odd activation function) in terms of input/output behavior has been previously examined by Sussmann [15] and Albertini-Sontag [1] for discrete-time and in [2] for continuous time. Second, as construed in this paper, observability of analog networks has been partially examined in [7], where it is shown that not all such discrete-time networks are observable. Observability of continuous-time analog networks remains a widely open problem.

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